Sturm Liouville Theory

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The problem

In this article we discuss some properties of the Sturm-Liouville (SL) system. This is a family of differential equations of the form

$$\partial_t[p(t)\partial_t u(t)] + [\lambda r(t) + q(t)]u(t) = 0 \tag{$*$}$$

on an interval $[a, b]$ with an initial condition of the form

$$\begin{aligned} \alpha u(a) + \tilde{\alpha} u'(a) &= 0 \\ \beta u(b) + \tilde{\beta} u'(b) &= 0 \end{aligned} \tag{$**$}$$

where

$$\alpha^2 + \tilde{\alpha}^2 \neq 0, \beta^2 + \tilde{\beta}^2 \neq 0$$

and λ is some parameter. Furthermore, we assume that $p > 0$ on $[a, b]$ and $r > 0$ at least on (a, b) with $p \in \mathcal{C}^1([a, b]), q, r \in \mathcal{C}^0([a, b])$. Note that when we talk about the derivative at the end points, we mean the one sidede derivative. This is the convention adopted in what follows.

What we would like to do is to find the values of the parameter λ for which $(*)$ has a (non trivial) solution with the initial conditions $(**)$. Those λ for which the system has a solution are called **eigenvalues** of the system and the corresponding solution is called the **eigenfunction**. The set of eigenvalues of the system is called the spectrum of the system.

If u is a solution to $(*)$, then we have

$$Lu = \lambda u$$

where L is the linear operator

$$L = \frac{1}{r(t)}[\partial_t[p(t)\partial_t] - q(t)].$$

This is why the λ are called eigenvalues, and the corresponding solutions eigenfunctions. We first find solutions to $(*)$ and later modify it to satisfy the initial conditions.

A special case

We will see what happens when p, q, r are constants. In this case, we have the auxillary equation (obtained by substituting $u = Ae^{kt}$)

$$pk^2 + (\lambda r + q) = 0$$

whose roots are $0, \pm\sqrt{-\frac{\lambda r + q}{p}}$. So, the solutions are either oscillatory or exponential.

Example: $u'' + \lambda u = 0$ on $[0, \pi]$ with $u(0) = u(\pi) = 0$.

1. $\lambda = 0$: u is linear, $u(0) = u(\pi) = 0 \Rightarrow u = 0$.
2. $\lambda < 0$: $u(t) = Ae^{\sqrt{-\lambda}t} + Be^{\sqrt{-\lambda}t}$, solving for initial conditions gives $A = B = 0$, so u is identically 0.
3. $\lambda > 0$: $u(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$. $u(0) = 0$ gives $A = 0$. If λ is a square integer there is no condition on B, else $B = 0$.

Prufer substitution

We do the following change of variables:

$$\begin{aligned} p(t)u'(t) &= R(t)\cos\theta(t) \\ u(t) &= R(t)\sin\theta(t) \end{aligned}$$

This substitution is called the Prufer substitution. Here $R(t)$ is called the amplitude and $\theta(t)$ is called the phase of the system.

Now if $R(t_0) = 0$ for some t_0, then $u(t_0) = u'(t_0) = 0$. Since the zero function is always a solution to $(*)$, by uniqueness (since p, q, r are smooth enough, uniqueness is guaranteed) u is the constant zero function. Since we are looking for non trivial solutions, it is safe to assume that R is never zero, in which case the substitution above is a valid one, i.e. has a reverse substitution.

We next obtain the equations satisfied by R, θ. Eliminating R, we have

$$\begin{aligned} \sin\theta(pu') &= \cos\theta(u) \\ \cos\theta(pu')\theta' + \sin\theta(pu')' &= -\sin\theta(u)\theta' + \cos\theta(u') \\ R(\cos^2\theta)\theta' - R\sin^2\theta(\lambda r + q) &= -R(\sin^2\theta)\theta' + \frac{1}{p}R\cos^2\theta \end{aligned}$$

Since R is never 0, we find that θ satisfies

$$\partial_t\theta(t;\lambda) = (\lambda r(t) + q(t))\sin^2\theta(t;\lambda) + \frac{1}{p(t)}\cos^2\theta(t;\lambda) \qquad (+)$$

Next, we have

$$\begin{aligned} R^2 &= (pu')^2 + u^2. \\ 2RR' &= 2(pu')(pu')' + 2uu' \\ &= 2R\cos\theta(R'\cos\theta - R\sin\theta\theta') + \frac{2}{p}R^2\cos\theta\sin\theta \\ RR' &= RR'\cos^2\theta - R^2\cos\theta\sin\theta\theta'\frac{1}{p}R^2\cos\theta\sin\theta \\ RR'\sin^2\theta &= R^2\cos\theta\sin\theta\left[-Q\sin^2\theta - \frac{1}{p}\cos^2\theta + \frac{1}{p}\right] \end{aligned}$$

where $Q = \lambda r + q$. So, R satisfies the equation

$$\partial_t R(t) = \frac{1}{2}\left[\frac{1}{p(t)} - \lambda r(t) - q(t)\right]R(t)\sin(2\theta(t;\lambda))$$

Does $(+)$ have a solution? Observe that since Q, p are continuous, p is positive on $[a, b]$, the derivative of θ is bounded as follows

$$\sup_{[a,b]}|\theta'| \leq \sup_{[a,b]}|Q| + \sup_{[a,b]}\left|\frac{1}{p}\right| < \infty.$$

So, with any initial condition on θ, there is a unique solution. Once θ is known, R is easily found by

$$R(t) = K\exp\left[\frac{1}{2}\int_a^t\left[\frac{1}{p(s)} - Q(s)\right]\sin(2\theta(s;\lambda))ds\right]$$

where K is some constant to be determined by initial conditions $(**)$. So, now the problem is to find and analyse the properties of solutions of $(+)$.

Prufer substitution helped us go from one second order equation to two (coupled) first order ones. However, knowledge of θ allows us to compute R. Under the assumption that our solution u is not identically 0, we have that R is never zero. So, u is zero only when θ is an integral multiple of π.

Lemma 1. *Consider the equation*

$$(pu')' + qu = 0$$

where $p \in \mathcal{C}^1([a,b]), q \in \mathcal{C}^0([a,b])$ and $p > 0$. The associated phase equation is

$$\theta' = q \sin^2 \theta + \frac{1}{p} \cos^2 \theta.$$

Given an integer m, there is at most one point c in $[a,b]$ where $\theta = m\pi$. Below c, $\theta(x) < m\pi$ and above strictly greater.

Proof. Observe that whenever $\theta(t) = m\pi$ for some integer, $\theta'(t) = \frac{1}{p} > 0$. So wherever θ hits a level $m\pi$, it has a positive slope. In fact, if $\theta(t) = m\pi$, then there is a neighbourhood of t where θ is strictly increasing. This follows by the continuity of θ'. Set $\theta^{-1}(\{m\pi\}) = A$.

Our next observation is that if $c_1, c_2 \in A$, say $c_1 < c_2$, then there's a neighbourhood $(c_1 \pm \epsilon_1)$ around c_1 and a $(c_2 \pm \epsilon_2)$ where θ is strictly increasing. Take points $c_1 < d_1 < c_1 + \epsilon_1$ and $c_2 - \epsilon_2 < d_2 < c_2$. Then $\theta(d_1) > m\pi > \theta(d_2)$. By intermediate value theorem there's a third point c_3 between d_1 and d_2 where $\theta(c_3) = m\pi$.

Now suppose $c_0 < c_1$ are points in A. Recursively obtain a sequence $c_1 > c_2 > \cdots > c_0$ all in A as observed above. The sequence $\{c_n\}_{n\geq 1}$ is decreasing and bounded below by c_0. Let $d = \inf_{n\geq 1} c_n$. d may or may not be c_0, but that doesn't matter. Since there is a sequence of c_is converging to d, $\theta(d) = m\pi$ by continuity. This means that there is an $\epsilon > 0$ such that θ is strictly increasing on $(d \pm \epsilon)$. However there is also some N large such that $d < c_N < d + \epsilon$. This is a contradiction.

Therfore there is at most one value where θ attains $m\pi$ for any integer m. Suppose $\theta(c) = m\pi$, then because the derivative is positive at c, $\theta < m\pi$ to the left of c and $\theta > m\pi$ to the right of c. For example, to the left of c, θ has to either stay above $m\pi$ or below. If it stays above, then one look at the definition of the derivative tells us that $\theta'(c) \leq 0$ which is a contradiction. □

Comparision Theorems

Theorem 1. (*Gronwall's inequality; integral form*) *Let $u\colon [a,b] \to \mathbb{R}$ be continuous non-negative and satisfy*

$$u(t) \leq A + \int_a^t B(s)u(s)ds, t \in [a,b] \tag{1}$$

where $A \geq 0$ and $B\colon [a,b] \to \mathbb{R}$ is continuous nonnegative. Then

$$u(t) \leq A \exp\left(\int_a^t B(s)ds\right) \forall\, t \in [a,b].$$

Proof. Assume $A > 0$. We have

$$\frac{d}{dt}\left[A + \int_a^t B(s)u(s)ds\right] = B(t)u(t) \leq B(t)\left[A + \int_a^t B(s)u(s)ds\right].$$

Since $A > 0, B, u \geq 0$, RHS is positive and we have

$$\frac{d}{dt} \ln(A + \int_a^t B(s)u(s)ds) \leq B(t).$$

Integration preserves inequalities, so integrating the above from a to t yields

$$\ln\left[A + \int_a^t B(s)u(s)ds\right] - \ln A \leq \int_a^t B(s)ds.$$

Rearranging this gives us

$$u(t) \leq A + \int_a^t B(s)u(s)ds \leq A \exp\left(\int_a^t B(s)ds\right) \forall\, t \in [a,b].$$

Note that this inequality is sharp as $u(t) = A\int_a^t B(s)ds$ satisfies (1). Furthermore, when $A = 0$, we can always take $A = \epsilon$, and then send ϵ to 0. $\square$

Theorem 2. (*Gronwall's inequality; differential form*) *Let* $u\colon [a,b] \to \mathbb{R}$ *be a differentiable funtion and suppose*

$$\partial_t u(t) \leq B(t)u(t) \tag{1}$$

where $B\colon [a,b] \to \mathbb{R}$ *is continuous. Then*

$$u(t) \leq u(a)\exp\bigg(\int_a^t B(s)ds\bigg)\,\forall\, t \in [a,b].$$

Proof. Intuitively, we can see that if we divide (1) by u and integrate, we arrive at the inequality stated, however we have to be careful about division by zero, change of signs. We go about this in the opposite direction. We define a function v on $[a,b]$ by

$$v(t) = u(t)\exp\bigg(-\int_a^t B(s)ds\bigg).$$

v is a differentiable function and satisfies

$$\partial_t v(t) = \partial_t u(t)\exp\bigg(-\int_a^t B(s)ds\bigg) - B(t)u(t)\exp\bigg(-\int_a^t B(s)ds\bigg) \leq 0.$$

So, v is a decreasing function (mean value theorem), which means that $v(t) \leq v(a)$ for every t. Rewriting this gives

$$u(t) \leq u(a)\exp\bigg(\int_a^t B(s)ds\bigg)\,\forall\, t \in [a,b].$$

$\square$

Remark. Note that in both of the proofs above, we needed continuity of B only to establish the existence of certain integrals. Furthermore, the differential form doesn't have any assumptions on the sign of u or B. Also, the differential form holds with the inequality reversed as well (identical proof, left to the reader).

Theorem 3. (*Comparision Principle*) *Let* $u, v\colon [a,b] \to \mathbb{R}$ *be two differentiable functions. Suppose* $F\colon I \times \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitz function. Suppose* u *and* v *obey the differential inequalities*

$$\partial_t u(t) \leq F(t, u(t)); \partial_t v(t) \geq F(t, v(t))$$

for all $t \in [a,b]$. *Then*

1. *if* $u(a) < v(a)$, *then* $u(t) < v(t)$ *for every* t.
2. *if* $u(a) \leq v(a)$, *then* $u(t) \leq v(t)$ *for every* t.

Proof. Firstly, we can assume F is Lipschitz with a constant $L > 0$. This is because I is compact, u, v are continuous on a compact interval. So, the images fo u, v lie on some rectangle with base $[a,b]$. This rectangle is a compact convex set and here locally Lipschitz is same as globally Lipschitz.

Suppose $u(t_*) = v(t_*)$ for some $t_* \in (a,b]$. Let t_* be the smallest such t_*. Why does it exist? Well, it is given by

$$t_* = \inf_{[a,b]}(u-v)^{-1}(\{0\}).$$

Infimum exists because we are assuming that the set is non empty and it is a subset of a compact interval. Secondly, for continuity reasons $u(t_*) = v(t_*)$. Note that $a < t_*$ by hypothesis.

On $[a, t_*)$, we must have $u < v$ because $u(a) < v(a)$ and because of the minimality of t_*.

Next, we observe that

$$\partial_t u(t) - \partial_t v(t) \leq F(t, u(t)) - F(t, v(t)) \leq L|u(t) - v(t)|.$$

Take any $a < c < t_*$. Then restricting everything to $[a, c]$, we have on $[a, c]$

$$\partial_t u(t) - \partial_t v(t) \leq -L(u(t) - v(t)).$$

Apply Gronwall's inequality to get

$$u(t) - v(t) \leq (u(a) - v(a)) \exp(-L(t-a)) \,\forall\, t \in [a, c].$$

In particular,

$$u(c) - v(c) \leq (u(a) - v(a))e^{-L(c-a)} < 0.$$

Now send c to t_*, then by squeeze theorem, we get

$$\lim_{c \to t_*} (u(a) - v(a))e^{-L(c-a)} = 0$$

which is absurd (it is crucial that $t_* < \infty$). Therefore, we must never have $u = v$ anywhere. This also gives that $u < v$ everywhere (intermediate value theorem).

Next we prove the second part. We can assume $u(a) = v(a)$. Suppose there was some $t^* > a$ such that $u(t^*) > v(t^*)$.

Part 1 tells that when there's a strict inequality on one end, then the same inequality holds to the right of the point. Here we would like to go to the left of t^* and arrive at a contradiction. How would we go to the left? Going left is the same as going to the right from "behind the graph". Moreover, note that the inequality has flipped, we have $u(t^*) > v(t^*)$. However, we need to proceed with caution, we want the differential inequalities to be satisfied.

We define $u_r, v_r \colon [0, 1] \to \mathbb{R}$ by

$$\begin{aligned} u_r(t) &= u(ta + (1-t)t^*) \\ v_r(t) &= v(ta + (1-t)t^*) \end{aligned}$$

We see whether the differential inequalities are satisfied

$$\partial_t u_r(t) = (a - t^*)\partial_t u(ta + (1-t)t^*) \geq (a - t^*)F(ta + (1-t)t^*, u_r(t))$$

because $a < t^*$.

We have

$$\begin{aligned} [0, 1] \times \mathbb{R} &\to [a, t^*] \times \mathbb{R} \to \mathbb{R} \\ (t, x) &\mapsto (at + (1-t)t^*, x) \mapsto F(at + (1-t)t^*, x) \end{aligned}$$

The first of these maps is a homeomorphism that at most scales the distance by a constant factor. Since F is locally Lipschitz, the composition is also locally Lipschitz, for given a point in $[0, 1] \times \mathbb{R}$, we look at its image, find a neighbourhood around it where F is Lipschitz with some constant, pull the neighbourhood back to $[0, 1] \times \mathbb{R}$, and the Lipschitz constant is scaled by at most a factor of $t^* - a$.

Call this new F as G. Then

$$\partial_t u_r(t) \geq (a - t^*)G(t, u_r(t));\ \partial_t v_r(t) \leq (a - t^*)G(t, v_r(t))$$

and $(a - t^*)G$ is also locally Lipschitz. Furthermore we have $u_r(0) > v_r(0)$, therefore by part 1, we have $u_r > v_r$ on $[0, 1]$. In particular, $u(a) = u_r(1) > v(a) = v_r(1)$ which is a contradiction.

Therefore $u \leq v$ on $[a, b]$. □

Corollary. *With the setup as above, if $u(a) \leq v(a)$ and if for some $c > a$, we have $u(c) = v(c)$, then $u = v$ on $[a, c]$.*

Remark. Sometimes we may have two functions u, v with

$$\partial_t u(t) = F(t, u(t)) \text{ and } \partial_t v(t) = G(t, u(t)).$$

If we know that $F(t, x) \leq G(t, x)$, then we can write $\partial_t v(t) \geq F(t, v(t))$ and apply the comparision principle.

Theorem 4. (*Sturm Comparision Principle*) *Suppose p_1, p_2, q_1, q_2 are function on $[a, b]$ with $p_1, p_2 \in \mathcal{C}^1([a, b])$ and $q_1, q_2 \in \mathcal{C}^0([a, b])$. Further assume that $0 < p_1 \leq p_2$ and $q_2 \leq q_1$. Consider the system*

$$(p_1 u')' + q_1 u = 0 \tag{a}$$

$$(p_2 v')' + q_2 v = 0 \tag{b}$$

If u, v are non trivial solutions of $(a), (b)$ respectively, then between any two zeros of v, there's at least one zero of u unless u is a multiple of v.

Proof. We look at the phase equations of these two equations. They are given as follows

$$\theta_1' = q_1 \sin^2 \theta_1 + \frac{1}{p_1} \cos^2 \theta_1 =: F_1(t, \theta_1(t)) \tag{a'}$$

$$\theta_2' = q_2 \sin^2 \theta_2 + \frac{1}{p_2} \cos^2 \theta_2 =: F_2(t, \theta_2(t)) \tag{b'}$$

The phase equations were derived by the Prufer substitutions. Since we are assuming that the solutions u, v are non zero, their zeros (i.e., points where they vanish) occur at points where their phase angles are integer multiples of π. By the assumptions on p_i, q_i, we have

$$F_1(t, x) \geq F_2(t, x)$$

for every $(t, x) \in [a, b] \times \mathbb{R}$. Furthermore, the trignometric functions $\sin, \cos$ are smooth, bounded, so they are Lipschitz. We have all the ingredients to apply comparision principles.

Assume $c_1 < c_2$ are two consecutive roots of v. By the lemma in the last section, we must have $\theta_2(c_1) = n\pi, \theta_2(c_2) = (n+1)\pi$ for some integer n. We restrict everything to the interval $[c_1, c_2]$. Observe that the solutions to the phase equation can be translated by mulitples of π, that is $\theta_1 + k\pi$ are all solutions to (a') as k varies over the integers. So, by adding a suitable multiple of π, we may assume that $\theta_1(c_1) \in [n\pi, (n+1)\pi)$.

Now, on $[c_1, c_2]$, we have

$$\partial_t \theta_1(t) = F_1(t, \theta_1(t))$$
$$\partial_t \theta_2(t) = F_2(t, \theta_2(t)) \leq F_1(t, \theta_2(t))$$

and $\theta_1(c_1) \geq \theta_2(c_1)$. Therfore, by comparision principle, we must have $\theta_1 \geq \theta_2$ on $[c_1, c_2]$.

If $\theta_1(c_2) > (n+1)\pi > \theta_1(c_1)$ and so there must be a point $x \in (c_1, c_2)$ where $\theta_1(x) = (n+1)\pi$ and hence there's a root of u between c_1, c_2.

Else $\theta_1(c_2) = (n+1)\pi = \theta_2(c_2)$, then the comparision principle gurantees that $\theta_1 = \theta_2 = \theta$ everywhere on $[c_1, c_2]$. Then it follows that $F_1(t, \theta(t)) - F_2(t, \theta(t)) = 0$ on $[c_1, c_2]$, i.e.,

$$(q_1(t) - q_2(t)) \sin^2 \theta(t) + (\frac{1}{p_1(t)} - \frac{1}{p_2(t)}) \cos^2 \theta(t) = 0 \, \forall \, t \in [c_1, c_2].$$

Since everything is positive, continuity gives $q_1 = q_2, p_1 = p_2$ on $[c_1, c_2]$. Further, since the amplitude R is determined upto a scalar multiple, it follows that u is a multiple of v on $[c_1, c_2]$. □

Back to SL systems

Now we go back to our analysis of $(*)$ with initial conditions $(**)$. We recall the equations here

$$\partial_t [p(t) \partial_t u(t)] + [\lambda r(t) + q(t)] u(t) = 0 \tag{$*$}$$

with initial conditions

$$\alpha u(a) + \tilde{\alpha} u'(a) = 0$$
$$\beta u(b) + \tilde{\beta} u'(b) = 0$$

where

$$\alpha^2 + \tilde{\alpha}^2 \neq 0, \beta^2 + \tilde{\beta}^2 \neq 0,$$

λ is some parameter and $p, r > 0$ on $[a, b]$ with $p \in \mathcal{C}^1([a, b]), q, r \in \mathcal{C}^0([a, b])$.

The Prufer substitution gives us the phase equation

$$\partial_t \theta(t; \lambda) = (\lambda r(t) + q(t)) \sin^2 \theta(t; \lambda) + \frac{1}{p(t)} \cos^2 \theta(t; \lambda) \tag{+}$$

and

$$\partial_t R(t) = \frac{1}{2}\left[\frac{1}{p(t)} - \lambda r(t) - q(t)\right] R(t) \sin(2\theta(t; \lambda)).$$

Now we want to study the eigenvalues, and the zeroes of a solution u to $(*)$. First we study what happens to the phase at a time t as λ varies. So far we haven't considered the initial condition. So, we set up the initial conditions as follows, let $\gamma \in [0, \pi)$ be such that

$$\alpha \sin \gamma + \tilde{\alpha} p(a) \cos \gamma = 0.$$

It exists because one of $\alpha, \tilde{\alpha}$ is non zero. Henceforth all θ solutions to $(+)$ are going to satisfy $\theta(a) = \gamma$.

Theorem 5. *For a fixed $t > a$, $\theta(t; \lambda)$ is a strictly increasing continuous function of λ*

Proof. Fix $\lambda_1 < \lambda_2$, real numbers. Let θ_1, θ_2 be the corresponding solutions to $(+)$ with initial condition $\theta_1(a) = \theta_2(a) = \gamma$.

Firstly, by the comparision principle, we have that $\theta_1 \leq \theta_2$ because $\theta_1(a) = \theta_2(a) = \gamma$. So, $\theta(t; \lambda)$ is increasing in λ. Furthermore, it is strictly increasing for if $\theta_1(t) = \theta_2(t)$ for some $t > a$, then we must have

$$\theta_1 = \theta_2 \text{ on } [a, t]$$

which would mean that the derivatives are equal, which is not true because $\lambda_1 < \lambda_2$. This conclusion follows because $r > 0$ on $(a, t]$ and $\sin^2$ is also non zero for at least one point here.

Next we have

$$\partial_t(\theta_1 - \theta_2) = (\lambda_1 \sin^2 \theta_1 - \lambda_2 \sin^2 \theta_2) r(t) + q(t)(\sin^2 \theta_1 - \sin^2 \theta_2) + \frac{1}{p}(\cos^2 \theta_1 - \cos^2 \theta_2).$$

Let r_M, q_M, p_m be the maximum, maximum, and minimum of r, q, p respectively on $[a, b]$. We know that $\sin^2, \cos^2$ are Lipschitz, since they have bounded derivatives, with some constants c_1, c_2. Combining these constants, we see that

$$|\partial_t(\theta_1 - \theta_2)(t)| \leq |\lambda_1 \sin^2 \theta_1(t) - \lambda_2 \sin^2 \theta_2(t)| r_M + C|\theta_1(t) - \theta_2(t)|$$

where C is some other constant dependeing on c_1, c_2, p_m, q_M.

To the first term, add and subtract a $\lambda_1 \sin^2 \theta_2(t)$, then we get

$$|\partial_t(\theta_1 - \theta_2)(t)| \leq c_1 r_M |\lambda_1(\theta_1(t) - \theta_2(t))| + |\lambda_1 - \lambda_2| r_M + C|\theta_1(t) - \theta_2(t)|.$$

Now let us fix λ_1 and vary λ_2. Since $\theta(t; \lambda)$ is strictly increasing with λ, $\theta(t; \lambda_1) - \theta(t; \lambda_2)$ has a fixed sign over $[a, b]$. So, depending on this sign, we can remove the modulus, and adjust the left side appropriately. So, if $\lambda_2 > \lambda_1$, we have

$$\partial_t(\theta_2 - \theta_1)(t) \leq (\lambda_2 - \lambda_1) r_M + C(\theta_2(t) - \theta_1(t)).$$

Note that because λ_1 is fixed, it is a constant. Things are positive on the right hand side and non zero, so we can divide and get

$$\int_a^t \frac{\partial_t y}{(\lambda_2 - \lambda_1) r_M + C y} \leq \int_a^t 1 = (t - a) < b - a$$

$$\frac{1}{C}\ln\left[\frac{(\lambda_2-\lambda_1)r_M+Cy(t)}{(\lambda_2-\lambda_1)r_M}\right]\leq(b-a)$$
$$(\lambda_2-\lambda_1)r_M+Cy(t)\leq(\lambda_2-\lambda_1)r_Me^{C(b-a)}$$
$$y(t)\leq\frac{1}{C}(\lambda_2-\lambda_1)r_M(e^{C(b-a)}-1)=(\lambda_2-\lambda_1)D$$

where $y(t)=\theta_2(t)-\theta_1(t)$ and D is the resulting constant. The integral works out to be of the form $\ln$ because $y\mapsto(\lambda_2-\lambda_1)r_M+Cy$ is a diffeomorphism, and change of variables applies. Had $\lambda_2<\lambda_1$, then working with $\theta_1-\theta_2$, we obtain

$$\theta_1(t)-\theta_2(t)\leq(\lambda_1-\lambda_2)D.$$

Eitherway it is clear that for any fixed $t>a$, $\theta(t;\lambda)$ is a continuous function of λ. □

Now we move onto the second initial condition. This condition is of the form

$$\beta u(b)+\tilde{\beta}u'(b)=0.$$

Use the Prufer substitution as before and find a $\delta\in(0,\pi]$ with

$$\beta\sin\delta+p(b)\tilde{\beta}\cos\delta=0.$$

We can translate δ by any $n\pi$, however note that δ must be greater than 0. This is because we have assumed $\theta(a)=\gamma\in[0,\pi)$. So, if θ attains 0 at some t, then by the remark earlier θ must be negative to the left of t contradicting $\theta(a)\geq 0$. So, we must have

$$\theta(b)=\delta+n\pi, n=0,1,\ldots$$

The question then is whether all possible n are realizable, i.e., are there eigenvalues λ_n with

$$\theta(b;\lambda_n)=\delta+n\pi.$$

At this point, we can say that the number of eigenvalues are at most countable because $\theta(b;\lambda)$ is strictly increasing with λ and hence, each $\delta+n\pi$ can be realised with at most one λ. Since $\theta(b;\lambda)$ is strictly increasing, there is hope that if λ is large enough, it can attain any value.

Suppose $\theta(b;\lambda)=\delta+n\pi>n\pi$. Then θ must attain the values $\pi,2\pi,\ldots,n\pi$, and possible 0 when $\gamma=0$. So u must have n zeroes. The next thing we do is to see whether this is possible. For $\lambda>0,\theta$ satisfies

$$\partial_t\theta=(\lambda r(t)+q(t))\sin^2\theta+\frac{1}{p(t)}\cos^2\theta\geq(\lambda r_m+q_m)\sin^2\theta+\frac{1}{p_M}\cos^2\theta$$

where m is for minimum and M for maximum over $[a,b]$. The expression on the right is the equation satisfied by the phase of the SL system

$$\partial_t(p_M\partial_t v)+(\lambda r_m+q_m)v=0$$

whose general solution is given by

$$v=c_1\exp\left(\sqrt{-\frac{\lambda r_m+q_m}{p_M}}t\right)+c_2\exp\left(-\sqrt{-\frac{\lambda r_m+q_m}{p_M}}t\right).$$

Sturm comparision theorem applies and we have that between any two zeroes of v there is at least one zero of u. It is important that $\lambda>0$, else the inequality above is not true.

Remark. Sturm comparision tells that either the zeroes are interspersed or the solutions are equal on the interval in question. Eitherways, between two zeroes of one solution, we will have a zero of the other and that is all we will require.

Remark. We don't need any information about the constants c_1,c_2 because all we are interested in are the zeroes. So, we can choose $c_1=1,c_2=0$. The Sturm comparision principle works for any solution of the differential equations no matter the initial condition.

Theorem 6. *For a fixed $t > a$, $\lim_{\lambda \to \infty} \theta(t; \lambda) = \infty$.*

Proof. Since we are dealing with $\lambda > 0$, the inequalities required for Sturm comparision principle hold and we can apply it with v as above. If λ is large enough that $\frac{\lambda r_m + q_m}{p_M} > 0$, then we can take $v = \sin \omega t$ whose zeroes are

$$\frac{n\pi}{\omega}, n \in \mathbb{Z}$$

where $\omega = \sqrt{\frac{\lambda r_m + q_m}{p_M}}$. For the moment assume $r_m > 0$. We are interested in the roots in $[a, b]$. So, the kth zero after a is

$$z_k = \frac{(\lceil \frac{a\omega}{\pi} \rceil + k - 1)\pi}{\omega}.$$

We have

$$a \leq z_k \leq a + \frac{k\pi}{\omega}.$$

For sufficiently large λ, we can make $a + \frac{k\pi}{\omega} < b$, which means that there must be at least $k - 1$ roots of u in $[a, a + \frac{k\pi}{\omega}]$. It follows that as $\lambda \to \infty$, the kth zero of u exists and converges to a. This is our first result:

$$\lim_{\lambda \to \infty} s_k = a$$

where s_k is the kth zero of u which exists for λ beyind some cut-off point.

Secondly, given a $t > a$, and N integer, for λ large such that

$$\frac{N\pi}{\omega} < t - a$$

the interval $[a, t]$ contains N zeroes of v, hence at least $N - 1$ zeroes of u. Lemma 1 above gives that $\theta(t; \lambda) > (N - 1)\pi$. So, as λ increases, $\theta(t; \lambda) \to \infty$.

If $r = 0$ at one of a, b then the above analysis fails. Take $\epsilon, \delta > 0$ and small enough and restrict everything to the interval $[a + \epsilon, b - \delta] = [a_1, b_1]$. Let w be the restriction of u to (a_1, b_1). The above analysis holds in this interval and it follows that w attains at least n roots in (a_1, b_1) and that the nth zero tends to a_1. This means that u must contain at least n roots as well. For $\lambda >> 0$, $s'_k < a_1 + \epsilon \Rightarrow s_k < a + 2\epsilon$ where s'_k is the kth root of w. This gives that the kth root os u satisfies $s_k < a + 2\epsilon$ because the kth root of u must appear before the kth root of w. Here ϵ can be chosen independent of λ, so $s_k \to a$. Note that as λ increases s'_k is the kth root of w, but it could be any s_n for $n > k$, because upon increasing λ, there may be an increase in the roots of u between a, a_1. □

Theorem 7. *For a fixed $t > a$, $\lim_{\lambda \to -\infty} \theta(t; \lambda) \to 0$.*

Proof. The following proof is taken from Lebovitz's book. We have

$$\partial_t \theta = (\lambda r(t) + q(t)) \sin^2 \theta + \frac{1}{p(t)} \cos^2 \theta \leq \lambda r(t) \sin^2 \theta + K.$$

Write $K = q_M + \frac{1}{p_m}$. The idea is as follows. If we can make the slope θ' very small, then θ must decrease rapidly, however since it cannot cross 0, and the interval we are working over is finite, θ must approach 0. To make the derivative small, we wouls like

$$\lambda r(t) \sin^2 \theta + K < -N$$

where N is some large positive number. In this case, we need

$$\lambda < -\frac{(N + K)}{r(t) \sin^2 \theta}.$$

We can replace $r(t)$ with r_m and maintain the inequality, but θ has a dependence on λ. One way to avoid this is to consider the minimum value $\sin^2$ attains, and to obtain this minimum we need to be clever and avoid $0, \pi$.

Now, fix $t > a$ and $\epsilon > 0$. First we consider $t \neq b$. Since $\theta' < K$, we have

$$\theta(x) \leq \gamma + K(x - a).$$

So, take $a_1 > a$ such that $\theta(a_1) < \gamma + \epsilon < \pi - \epsilon$ with the assumption that ϵ is small enough so that $\gamma + 2\epsilon < \pi$.

Next consider the line L from $(a_1, \gamma + \epsilon)$ to (t, ϵ). L has slope

$$m = -\frac{\gamma}{t - a_1}.$$

We take this as our $-N$ and set

$$\Lambda = \frac{m - K}{r_m \sin^2 \epsilon}$$

where $r_m > 0$ is the minimum of r on $[a_1, t]$.

At a_1, we have $\theta(a_1) < L(a_1)$. Suppose $\theta > L$ at somewhere in $[a_1, t]$, then it θ and L must meet at a first point t_* (the first point exists by taking infimum and for continuity reasons). At t_*, we have $\theta(t_*) \in [\epsilon, \pi - \epsilon]$ and

$$\theta'(t; \lambda) \leq \lambda r_m \sin^2 \epsilon + K < m$$

where $r_m > 0$ is the minimum of r in $[a_1, t]$ and $\lambda < \Lambda$. However since t_* is te first point where θ meets L, we must have $\theta'(t_*) > m$ which is a contradiction. Therefore θ must lie below L, and in particular $\theta(t; \lambda) < \epsilon$ for $\lambda < \Lambda$.

Thus, given $\epsilon > 0$, we can choose a_1 independent of λ, and obtain Λ, we have for $\lambda < \Lambda$, $\theta(t; \lambda) < \epsilon$. Therefore $\theta(t; \lambda) \to 0$ as $\lambda \to -\infty$ for $t \in (a, b)$.

Now for $t = b$. The analysis above works verbatim if $r(b) \neq 0$. Else, we choose a $b_1 < b$ depending on ϵ such that $\theta(b_1) > \theta(b) + \epsilon$. This can be done because $\theta' < K$ which gives $\theta(b) - \theta(b_1) < K(b - b_1)$ which is a condition independent of λ. There is a Λ such that for $\lambda < \Lambda$, we have $\theta(b_1; \lambda) < \epsilon$ which gives $\theta(b; \lambda) < 2\epsilon$. Thus, $\theta(b; \lambda) \to 0$ as $\lambda \to -\infty$. □

Finally, we have the theorem we have been working towards.

Theorem 8. (*Sturm-Liouville theorem*) *The boundary value problem* $(*), (**)$ *has an infinite sequence of eigenvalues* $\{\lambda_n\}_0^\infty$ *with* $\lambda_n < \lambda_{n+1}$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$. *The eigenfunction* u_n *associated to* λ_n *has precisely* n *zeroes in* (a, b).

Proof. We know that $\theta(b; \lambda)$ is a continuous, strictly increasing function of λ and $\lim_{\lambda \to -\infty} \theta(b; \lambda) = 0, \lim_{\lambda \to \infty} \theta(b; \lambda) = \infty$. By continuity, there is a λ_n such that $\theta(b; \lambda_n) = \delta + n\pi$, for $n = 0, 1, \ldots$. These are precisely the eigenvalues of the system, and the other properties are easily verified. □

Conclusion

We have proved the Sturm-Liouville theorem. There are further modifications of the system $(*)$. One modification is when r is allowed to change signs. We have considered the case when r is strictly positive on (a, b). In a more general case, for example when $r > 0$ on (a, x_*) and $r < 0$ on (x_*, b), what we can do is to analyse the problem on each subinterval. It turns out that in this case too, there are an infinite number of eigenvalues, going to $+\infty$ and $-\infty$.

Another modification is when p, r, q are periodic. These modifications can be found in the chapter *Oscillation Theory* from the book *Ordinary Differential Equations* by Norman R. Lebovitz listed in the references.

References

[1] These notes were partly based on the Differential Equations course taught by Prof. Matthew Joseph at ISI Bangalore.

[2] Partly based on the chapter *Oscillation Theory* from the book *Ordinary Differential Equations* by Norman R. Lebovitz http://people.cs.uchicago.edu/ lebovitz/odes.html

[3] Some statements of theorems were picked from *Nonlinear dispersive equations: local and global analysis* by Terence Tao